Maxwell's Equations

In 1860, while Americans were waging a bloody civil war, a "thorough old Scotch laird" (then only 29) named James Clerk Maxwell was assembling the known laws of electromagnetism into a compact and elegant form that was to lead, a year later, to the discovery that *light* is in fact a propagating disturbance in the electromagnetic fields. That discovery was later to overturn all the conceptual foundations of classical Physics and leave "common sense" in much the same condition as the United States after the Civil War. It was hard times all around, but *exciting*....

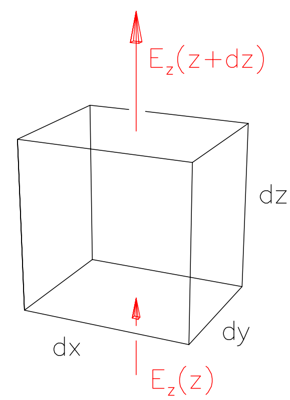

Figure 22.1 An infinitesimal volume of space.

22.1 Gauss' Law

By now you are familiar with GAUSS' LAW in its integral form,

$$\epsilon_\circ \oiint_{\mathcal{S}} \vec{E} \cdot d\vec{A} = Q_{\text{encl}} \qquad (1)$$

where Q_{encl} is the electric charge enclosed within the closed surface $\mathcal{S}$. Except for the "fudge factor" $\epsilon_\circ$, which is just there to make the units come out right, GAUSS' LAW is just a simple statement that electric field "lines" are continuous except when they start or stop *on electric charges*. In the absence of "sources" (positive charges) or "sinks" (negative charges), electric field lines obey the simple rule, "What goes in must come out." This is what GAUSS' LAW says.

There is also a GAUSS' LAW for the *magnetic* field $\vec{B}$; we can write it the same way,

$$(\text{some constant}) \oiint_{\mathcal{S}} \vec{B} \cdot d\vec{A} = Q_{\text{Magn}} \qquad (2)$$

where in this case Q_{Magn} refers to the enclosed *magnetic charges*, of which (so far) none have ever been found! So GAUSS' LAW FOR MAGNETISM is usually written with a zero on the right-hand side of the equation, even though no one is very happy with this lack of symmetry between the electric and magnetic versions.

Suppose now we apply GAUSS' LAW to a small rectangular region of space where the z axis is chosen to be in the direction of the electric field, as shown in Fig. 22.1.[1] The flux of electric field *into* this volume at the bottom is $E_z(z)\,dx\,dy$. The flux *out* at the top is $E_z(z+dz)\,dx\,dy$; so the *net* flux *out* is just $[E_z(z+dz) - E_z(z)]\,dx\,dy$. The definition of the *derivative of E with respect to z* gives us $[E_z(z+dz) - E_z(z)] = (\partial E_z/\partial z)\,dz$ where the partial derivative is used in acknowledgement of the possibility that E_z may also vary with x and/or y. GAUSS' LAW then reads $\epsilon_\circ(\partial E_z/\partial z)\,dx\,dy\,dz = Q_{\text{encl}}$. What is Q_{encl}? Well, in such a small region there is some approximately constant *charge density* ρ (charge per unit volume) and the volume of this region is $dV = dx\,dy\,dz$, so GAUSS' LAW reads $\epsilon_\circ(\partial E_z/\partial z)\,dV = \rho\,dV$ or just $\epsilon_\circ\,\partial E_z/\partial z = \rho$. If we now allow for the possibility of electric flux entering and exiting through the other faces (*i.e.* $\vec{E}$ may also have x and/or y components), perfectly analogous arguments hold for those components, with the resultant "outflow-ness" given by

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \vec{\nabla} \cdot \vec{E} \equiv \text{div}\,\vec{E}$$

where the GRADIENT operator $\vec{\nabla}$ is shown in its cartesian representation (in rectangular coordinates x, y, z). It has completely equivalent representations in other coordinate systems such as spherical (r, θ, ϕ) or cylindrical coordinates, but for illustration purposes the cartesian coordinates are simplest.

We are now ready to write GAUSS' LAW in its compact *differential* form,

$$\boxed{\epsilon_\circ\,\vec{\nabla} \cdot \vec{E} = \rho} \qquad (3)$$

and for the magnetic field, assuming no magnetic charges (MONOPOLES),

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0} \qquad (4)$$

These are the first two of MAXWELL'S EQUATIONS.

[1] This Figure is very similar to the one used to derive the EQUATION OF CONTINUITY, which in fact expresses the same basic principles (conservation of some "stuff" produced locally), although it is generally used for different purposes.

22.2 Faraday's Law

You should now be familiar with the long *integral* mathematical form of FARADAY'S LAW of MAGNETIC INDUCTION: in SI units,

$$\oint_{\mathcal{C}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{\ell}} \;=\; -\frac{\partial}{\partial t} \iint_{\mathcal{S}} \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{S}} \tag{5}$$

where the line integral of $\vec{\boldsymbol{E}}$ around the closed loop $\mathcal{C}$ is (by definition) the *induced* $\mathcal{EMF}$ around the loop and the right hand side refers to the *rate of change* of the *magnetic flux* through the area $\mathcal{S}$ *bounded* by that closed loop.

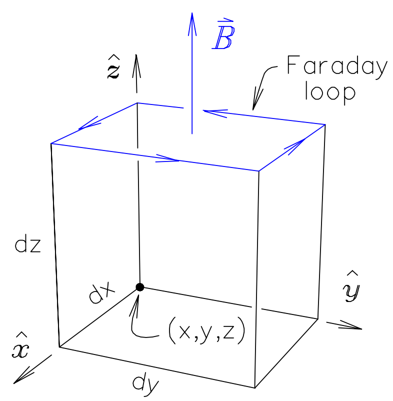

Figure 22.2 Another infinitesimal volume of space.

To make this easy to visualize, let's again draw an infinitesimal rectangular box with the z axis along the direction of the magnetic field, which can be considered more or less uniform over such a small region. Then the flux through the "Faraday loop" is just $B_z\,dx\,dy$ and the line integral of the electric field is

$$E_x(y)\,dx + E_y(x+dx)\,dy - E_x(y+dy)\,dx - E_y(x)\,dy.$$

(Yes it is. Study the diagram!) Here, as before, $E_y(x+dx)$ denotes the magnitude of the y component of $\vec{\boldsymbol{E}}$ along the front edge of the box, and so on. As before, we note that $[E_y(x+dx) - E_y(x)] = (\partial E_y/\partial x)\,dx$ and $[E_x(y+dy) - E_x(y)] = (\partial E_x/\partial y)\,dy$ so that FARADAY'S LAW reads

$$\left(\frac{\partial E_y}{\partial x}\,dx\right)\,dy \;-\; \left(\frac{\partial E_x}{\partial y}\,dy\right)\,dx \;=\; -\left(\frac{\partial B_z}{\partial t}\right)\,dx\,dy$$

which reduces to the *local* relationship

$$\left(\frac{\partial E_y}{\partial x} \;-\; \frac{\partial E_x}{\partial y}\right) \;=\; -\left(\frac{\partial B_z}{\partial t}\right)$$

between the "swirlyness" of the spatial dependence of the electric field and the rate of change of the magnetic field with time.

If you have studied the definition of the CURL of a vector field, you may recognize the left-hand side of the last equation as the z component of

$$\begin{aligned} \textbf{curl}\,\vec{\boldsymbol{E}} \;&\equiv\; \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} \\ &\equiv\; \hat{\boldsymbol{\imath}}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) \\ &+\; \hat{\boldsymbol{\jmath}}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) \\ &+\; \hat{\boldsymbol{k}}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right). \end{aligned}$$

The x and y components of $\textbf{curl}\,\vec{\boldsymbol{E}}$ are related to the corresponding components of $\partial\vec{\boldsymbol{B}}/\partial t$ in exactly the same way, allowing us to write FARADAY'S LAW in a *differential form* which describes part of the behaviour of electric and magnetic fields at every point in space:

$$\boxed{\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} \;=\; -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}} \tag{6}$$

This says, in essence, that *any change in the magnetic field with time induces an electric field perpendicular to the changing magnetic field.* Hold that thought.

22.3 Ampère's Law

You are probably also adept at using the trick developed by Henri Ampère for calculating the magnetic field ($\vec{\boldsymbol{H}} \equiv \vec{\boldsymbol{B}}/\mu$) due to various symmetrical arrangements of electric current (I). In its integral form and SI units, AMPÈRE'S LAW reads

$$\oint_{\mathcal{C}} \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{\ell}} \;=\; I + \frac{\partial}{\partial t} \iint_{\mathcal{S}} \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{S}} \tag{7}$$

where Maxwell's "DISPLACEMENT CURRENT" associated with a time-varying electric displacement $\vec{\boldsymbol{D}} \equiv \epsilon\vec{\boldsymbol{E}}$ has been included. This equation says (sort of), "The *circulation* of the magnetic field around a *closed loop* is equal to a constant times the total electric current *linking* that loop, *except when there is a changing electric field* in the same region."

As you know, this "Law" is used with various symmetry arguments to "finesse" the evaluation of magnetic fields due to arrangements of electric currents, much as GAUSS' LAW was used to calculate electric fields due to different arrangements of electric charges. Skipping over the details, let me draw your attention to the formal similarity to FARADAY'S LAW and state (this time without showing the derivation) that there is an analogous *differential form* of AMPÈRE'S LAW describing the behaviour of the fields at any point in space:

$$\boxed{\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{H}} \;=\; \vec{\boldsymbol{J}} \;+\; \frac{\partial \vec{\boldsymbol{D}}}{\partial t}} \tag{8}$$

If we ignore the current density $\vec{J}$ then this equation says (sort of), "*A changing electric field generates a magnetic field at right angles to it,*" which is rather reminiscent of what FARADAY'S LAW said.

Now we're getting somewhere.

22.4 Maxwell's Equations

In 1865, James Clerk Maxwell assembled all the known "Laws" of $\mathcal{E}\&\mathcal{M}$ in their most compact, elegant (differential) form, shown here in SI units:

GAUSS' LAW FOR ELECTROSTATICS:

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{9}$$

GAUSS' LAW FOR MAGNETOSTATICS:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{10}$$

FARADAY'S LAW:

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{11}$$

AMPÈRE'S LAW:

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \tag{12}$$

These four basic equations are known collectively as MAXWELL'S EQUATIONS; they are considered by most Physicists to be a beautifully concise summary of $\mathcal{E}\&\mathcal{M}$ phenomenology.

Well, actually, a *complete* description of $\mathcal{E}\&\mathcal{M}$ also requires two additional laws:

EQUATION OF CONTINUITY:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J} \tag{13}$$

LORENTZ FORCE:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right). \tag{14}$$

22.5 The Wave Equation

The two "Laws" of ELECTRODYNAMICS — FARADAY'S LAW and AMPÈRE'S LAW — can be combined to produce a very important result.

First let's simplify matters by considering the behaviour of electromagnetic fields in *empty space*, where

$$\rho = 0, \qquad \vec{J} = 0, \qquad \vec{D} = \epsilon_\circ \vec{E} \quad \text{and} \quad \vec{B} = \mu_\circ \vec{H}.$$

Our two equations then read

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{and} \quad \frac{\vec{\nabla} \times \vec{B}}{\mu_\circ} = \epsilon_\circ \frac{\partial \vec{E}}{\partial t}.$$

We can simplify further by assuming that the electric field is in the $\hat{y}$ direction and the magnetic field is in the $\hat{z}$ direction. In that case,

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \quad \text{and} \quad \frac{\partial B}{\partial x} = -\epsilon_\circ \mu_\circ \frac{\partial E}{\partial t}$$

where the second equation has been multiplied through by $\mu_\circ$.

If we now take the derivative of the first equation with respect to x and derivative of the second equation with respect to t, we get

$$\frac{\partial^2 E}{\partial x^2} = -\frac{\partial^2 B}{\partial x \partial t} \quad \text{and} \quad \frac{\partial^2 B}{\partial t \partial x} = -\epsilon_\circ \mu_\circ \frac{\partial^2 E}{\partial^2 t}.$$

$$\text{Since} \qquad \frac{\partial^2 B}{\partial x \partial t} = \frac{\partial^2 B}{\partial t \partial x},$$

the combination of these two equations yields

$$\frac{\partial^2 E}{\partial x^2} = \epsilon_\circ \mu_\circ \frac{\partial^2 E}{\partial^2 t}$$

which the discerning reader will recognize as the one-dimensional WAVE EQUATION for E,

$$\boxed{\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial^2 t} = 0 \qquad \text{where} \qquad c = \frac{1}{\sqrt{\epsilon_\circ \mu_\circ}}}$$

is the propagation velocity. You can easily show that there is an identical equation for B. A more general derivation yields the 3-dimensional version,

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{or} \qquad \Box^2 \vec{E} = 0$$

In either form, this equation expresses the fact that, since a changing electric field generates a magnetic field but that change in the magnetic field generates, in turn, an electric field, and so on, we can conclude that *electromagnetic fields will propagate spontaneously through regions of total vacuum* in the form of a *wave* of $\vec{E}$ and $\vec{B}$ fields working with/against each other.

This startling conclusion (in 1865) led to the revision of all the "classical" paradigms of Physics, even such fundamental concepts as space and time.