Electromagnetism

As suggested in the previous Chapter, Electricity and Magnetism (or $\mathcal{E}\&\mathcal{M}$, as they are known in the trade) are "weird" phenomena because the palpable *forces* they generate on objects seem to come from nowhere — nothing is "touching" the objects and yet they are moved. The related fact that we are unable to wilfully exert significant electrical or magnetic forces directly on objects around us using any combination of muscles or mechanical devices removes $\mathcal{E}\&\mathcal{M}$ still further from our personal sensory experience and thus makes them seem "weirder." Even the most seasoned $\mathcal{E}\&\mathcal{M}$ veteran still experiences a sense of primitive wonder when a magnet on top of the table moves "by magic" under the influence of another magnet underneath the table.

On the one hand, this makes $\mathcal{E}\&\mathcal{M}$ a fun subject to study. On the other hand, it makes $\mathcal{E}\&\mathcal{M}$ hard to teach, because it will never make "common sense" like nuts-and-bolts Mechanics. *C'est la vie.* As our first foray into "Weird Science" it is only fitting that $\mathcal{E}\&\mathcal{M}$ should be something we know is there but that we will just have to get used to instead of ever hoping to rectify it with our common sense. It is, of course, "common sense" itself that is defective....

17.1 "Direct" Force Laws

There are two fundamental kinds of forces in $\mathcal{E}\&\mathcal{M}$: the *electrostatic* force between two *charges* and the *magnetic* force between two *currents*. Let's start with the easy one.

17.1.1 The Electrostatic Force

First, what is a *charge*? We don't know! But then, we don't know what a *mass* is, either, except in terms of its behaviour: a mass resists acceleration by forces and attracts other masses with a gravitational force. The analogy is apt, in the sense that electrical charges exert forces on each other in almost exactly the same way as masses do, except for two minor differences, which I will come to shortly. Recall Newton's UNIVERSAL LAW OF GRAVITATION in its most democratic form: the force $\vec{F}^G_{12}$ acting *on* body #2 (mass m_2) *due to* body #1 (mass m_1) is

$$\vec{F}^G_{12} = -G \, \frac{m_1 m_2}{r_{12}^2} \, \hat{r}_{12}$$

where G is the Universal Gravitational Constant, r_{12} is the distance between the two masses and $\hat{r}_{12}$ is the unit vector pointing *from* #1 *to* #2. The *electrostatic* force $\vec{F}^E_{12}$ between two *charges* q_1 and q_2 is of exactly the same form:

$$\vec{F}^E_{12} = k_E \, \frac{q_1 q_2}{r_{12}^2} \, \hat{r}_{12} \qquad (1)$$

where k_E is some constant to make all the units come out right [allow me to sidestep this can of worms for now!]. Simple, eh? This force law, also known as the COULOMB FORCE,[1] has almost the same qualitative earmarks as the force of gravity: the force is "central" — *i.e.* it acts along the line joining the centres of the charges — and drops off as the inverse square of the distance between them; it is also proportional to each of the charges involved. [We could think of *mass* as a sort of "gravitational charge" in this context.]

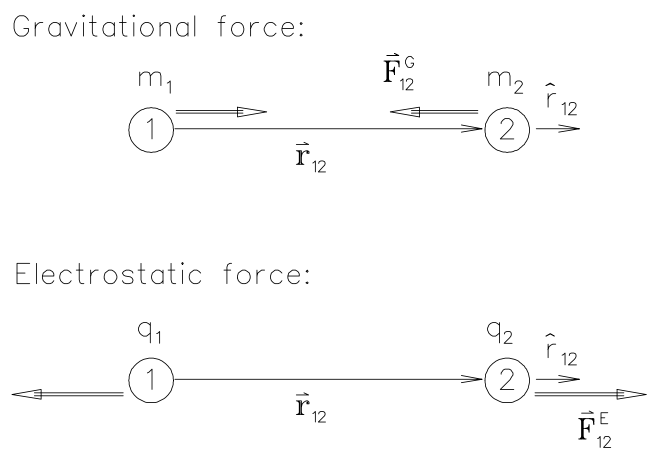

Figure 17.1 Comparison of the gravitational force $\vec{F}^G_{12}$ on mass m_2 due to mass m_1 and the electrostatic (Coulomb) force $\vec{F}^E_{12}$ on electric charge q_2 due to charge q_1.

So what are the "minor differences?" Well, the first one is in the *sign*. Both "coupling constants" (G and k_E) are defined to be *positive*; therefore the $-$ sign in the first equation tells us that the gravitational force $\vec{F}^G_{12}$ on mass #2 is in the *opposite direction* from the unit vector $\hat{r}_{12}$ pointing form #1 to #2 — *i.e.* the force is *attractive*, back toward the other body. All masses attract all other masses gravitationally; there are (so far as we know) no repulsive forces in gravity. Another way of putting it would be to say that "there are no negative masses." By contrast, electric charges come in both positive (+) and

[1] The COULOMB FORCE law, like the "coulomb" unit for electric charge (to be discussed later), is named after a guy called Coulomb; $\mathcal{E}\&\mathcal{M}$ units are littered with the names of the people who invented them or discovered related phenomena. Generally I find this sort of un-mnemonic naming scheme counterdidactic, but since we have no experiential referents in $\mathcal{E}\&\mathcal{M}$ it's as good a scheme as any.

negative (−) varieties; moreover, Eq. (1) tells us that the electrical force $\vec{F}_{12}^{E}$ on charge #2 is in the *same* direction as $\hat{r}_{12}$ as long as the product $q_1 q_2$ is positive — *i.e.*

> charges of *like* sign [*both* + or *both* −] *repel* whereas *unlike* charges *attract*.

This means that a positive charge and a negative charge of equal magnitude will get pulled together until their net charge is zero, whereupon they "neutralize" each other and cease interacting with all *other* charges. To a good approximation, this is just what happens! Most macroscopic matter is electrically *neutral*, meaning that it has the positive and negative charges pretty much piled on top of each other.[2]

The second "minor difference" between electrical and gravitational forces is in their *magnitudes*. Of course, each depends on the size of the "coupling constant" [G for gravity *vs.* k_E for electrostatics] as well as the sizes of the "sources" [m_1 and m_2 for gravity *vs.* q_1 and q_2 for electrostatics] so any discussion of magnitude has to be in reference to "typical" examples. Let's choose the heaviest stable elementary particle that has both charge and mass: the *proton*, which constitutes the nucleus of a hydrogen atom.[3] A proton has a positive charge of

$$e = 1.60217733(49) \times 10^{-19} \text{ coulomb (abbreviated C)} \tag{2}$$

[Don't worry about what a coulomb is just yet.] and a mass of

$$m_p = 1.6726231(10) \times 10^{-27} \text{ kg} \tag{3}$$

For *any* separation distance r, two protons *attract* each other (gravitationally) with a force whose magnitude F_G is $\frac{G\, m_p^2}{k_E\, e^2}$ times the magnitude F_E of the (electrostatic) force with which they *repel* each other. This ratio has an astonishing value of 0.80915×10^{-36} — the gravitational attraction between the two protons is roughly a *trillion trillion trillion* times *weaker* than the electrostatic repulsion. The electrical force wins, hands down. However, *in spite of its phenomenal puniness, gravity can overcome all other forces if enough mass gets piled up in one place.* This feature will be discussed at length later on, but for now it is time to discuss the basic *magnetic* force.

[2]On a *microscopic* scale there are serious problems with this picture. As the two charges get closer together, the force grows bigger and bigger and the *work* required to pull them apart grows without limit; in principle, according to Classical Electrodynamics, an infinite amount of work can be performed by two opposite charges that are allowed to "fall into" each other, providing we can set up a tiny system of levers and pulleys. Worse yet, the "self energy" of a *single* charge of vanishingly small size becomes infinite in the classical limit. But I am getting ahead of myself again....

[3]Now I am '*way* ahead of myself; but we do need something for an example here!

17.1.2 The Magnetic Force

As we shall see later, the "Laws" of $\mathcal{E}\&\mathcal{M}$ are so symmetric between electrical and magnetic phenomena that most Physicists are extremely frustrated by the fact that no one has ever been able to conclusively demonstrate the existence (other than theoretical) of a "magnetic charge" (also known as a *magnetic monopole*). If there were magnetic charges, the magnetic force equation would look just like the gravitational and electrostatic force laws above and this part of the description would be nice and simple. Alas, this is not the case. Static (constant in time) magnetic phenomena are generated instead by the steady *motion* of electric charges, represented by a *current* I (the charge passing some fixed point per unit time) in some direction $\vec{\ell}$. Usually (at least at the outset) we talk about currents flowing in a *conductor* (*e.g.* a wire) through which the charges are free to move with minimal resistance. Then $\vec{\ell}$ is a vector length pointing along the wire, or (if the wire is curved) $d\vec{\ell}$ is an infinitesimal *element* of the wire at some point. We may then think in terms of a "current element" $I d\vec{\ell}$.

One such current element $I_1 d\vec{\ell}_1$ exerts a magnetic force $d\vec{F}_{12}^{M}$ on a second current element $I_2 d\vec{\ell}_2$ at a distance $\vec{r}_{12}$ (the vector from #1 to #2) given by

$$d\vec{F}_{12}^{M} \;=\; k_M \, \frac{I_1 I_2}{r_{12}^2} \, d\vec{\ell}_2 \times (d\vec{\ell}_1 \times \hat{r}_{12}) \tag{4}$$

where k_M is yet another unspecified constant to make all the units come out right [just wait!] and again $\hat{r}_{12}$ is the *unit vector* defining the direction *from* #1 *to* #2.

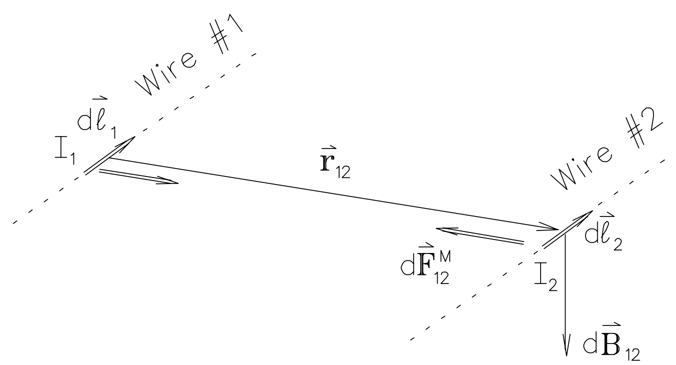

Figure 17.2 The magnetic force $d\vec{F}_{12}^{M}$ on current element $I_2 d\vec{\ell}_2$ due to current element $I_1 d\vec{\ell}_1$.

This ugly equation (4) does give us some important qualitative hints about the force between two current-carrying wires: the force between any two *elements* of wire drops off as the inverse square of the distance between them, just like the gravitational and electrostatic forces [although this isn't much use in guessing the force between real current-carrying wires, which don't come in infinitesimal lengths] and the force is in a direction perpendicular to both wires. In fact, if we are patient we can see which way the magnetic forces will act between two *parallel* wires: we can visualize a distance vector $\vec{r}$ from the first wire (#1) over to the second wire (#2); let it be perpendicular to both for convenience. The "RIGHT-HAND RULE" will then tell us the *direction* of $(d\vec{\ell}_1 \times \hat{r}_{12})$: if we "turn the screw" in the sense of cranking through the angle *from* $d\vec{\ell}_1$ *to* $\hat{r}_{12}$), a right-handed screw [the conventional kind] would move in the direction labelled $d\vec{B}_{12}$ in Fig. 17.2. This is the direction of $(d\vec{\ell}_1 \times \hat{r}_{12})$. Now if we crank $d\vec{\ell}_2$ into $d\vec{B}_{12}$, the turn of the screw will cause it to head back toward the first wire! Simple, eh?

Seriously, *no one* is particularly enthused over this equation! All anyone really retains from this intricate exercise is the following pair of useful rules:

1. Two parallel wires with electrical currents flowing in the *same* direction will *attract* each other.
2. Two parallel wires with electrical currents flowing in *opposite* directions will *repel* each other.

Nevertheless, electrical engineers and designers of electric motors and generators need to know just what sorts of forces are exerted by one complicated arrangement of current-carrying wires on another; moreover, once it had been discovered that moving charges create this weird sort of action-at-a-distance, no one wanted to just give up in disgust and walk away from it. What can we possibly do to make magnetic calculations manageable? Better yet, is there any way to make this seem more *simple*?

17.2 Fields

In Classical Mechanics we found several conceptual aids that not only made calculations easier by skipping over inessential details but also made it possible to carry around the bare essence of Mechanics in our heads in a small number of compact "Laws." This is generally regarded as a good thing, although of course we pay a price for every entrenched paradigm — we may lose the ability (if we ever had it!) to "see things as they are" without filtering our experience through models. I will leave that debate to the philosophers, psychologists and mystics; it is true even in Physics, however, that the more successful the paradigm the bigger the blind spot it creates for alternative descriptions of the same phenomena. This bothers most Physicists, too, but there doesn't seem to be a practical alternative; so we content ourselves with maintaining an awareness of our own systematic prejudices.

Perhaps the best example of this from the days of "Classical" Physics [*i.e.* before Relativity and Quantum Mechanics rained confusion down on all of us] is the invention of the ELECTRIC and MAGNETIC FIELDS, written $\vec{E}$ and $\vec{B}$, respectively. The idea of FIELDS is to break down the nasty problems described in the previous Section into two easier parts:

1. First, calculate the FIELD due to the *source* charge or current.
2. Then calculate the *force* on a *test* charge or current *due to* that FIELD.

This also makes it a lot easier to organize our calculations in cases where the *sources* are complicated arrays of charges and/or currents. Here's how it works:

17.2.1 The Electric Field

The ELECTRIC FIELD $\vec{E}$ at any point in space is defined to be the *force per unit test charge* due to all the other charges in the universe. That is, there is probably no "test charge" q there to experience any force, but *if there were* it would experience a force

$$\vec{F}_E = q\,\vec{E} \tag{5}$$

Note that since the force is a vector, $\vec{E}$ is a *vector field.*

Since by definition $\vec{E}$ is there even if there *isn't* any test charge present, it follows that there is an electric field at every point in space, all the time! [It might be pretty close to zero, but it's still there!][4]

Is the ELECTRIC FIELD real? No. Yes. You decide.[5] This paradigm makes everything so much easier that most Physicists can't imagine thinking about $\mathcal{E}\&\mathcal{M}$ any other way. Does this blind us to other possibilities? Undoubtedly.

A single isolated electric "source" charge Q [I am labelling it differently from my "test" charge q just to avoid confusion. Probably it won't work.] generates a *spherically symmetric* electric field

$$\vec{E} = k_E \, \frac{Q}{r^2} \, \hat{r} \tag{6}$$

[4] We often try to represent this graphically by drawing "lines of force" that show which way $\vec{E}$ points at various positions; unfortunately it is difficult to draw in $\vec{E}$ at *all* points in space. I will discuss this some more in a later Section.

[5] Define "real."

at any point in space specified by the vector distance $\vec{r}$ from Q to that point. That is, the field $\vec{E}$ is *radial* [in the direction of the radius vector] and has the same *magnitude* E at all points on an imaginary spherical surface a distance r from Q.

It might be helpful to picture the *acceleration of gravity* as a similar *vector field*:

$$\vec{g} \;=\; -G\,\frac{M_E}{r^2}\,\hat{r} \tag{7}$$

— *i.e.* $\vec{g}$ always points back toward the centre of the Earth (mass M_E) and drops off as the inverse square of the distance r from the centre of the Earth.

17.2.2 The Magnetic Field

Any current element $Id\vec{\ell}$ contributes $d\vec{B}$ to the magnetic field $\vec{B}$ at a given point in space:

$$d\vec{B} \;=\; k_M\,\frac{I\,d\vec{\ell}\times\hat{r}}{r^2} \tag{8}$$

where $\hat{r}$ is the unit vector in the direction of $\vec{r}$, the vector distance from the current element to the point in space where the magnetic field is being evaluated. Eq. (8) is known as the LAW OF BIOT AND SAVART. It is still not perfectly transparent, I'm sure you will agree, but it beats Eq. (4)!

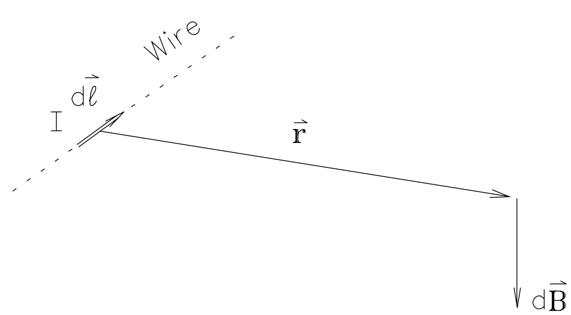

Figure 17.3 The magnetic field $d\vec{B}$ at position $\vec{r}$ due to a current element $Id\vec{\ell}$ at the origin.

17.2.3 Superposition

While it may seem obvious, it bears saying that the electric fields due to several different "source" charges or the magnetic fields due to several different "source" current elements are just added together (vectorially, of course) to make the net $\vec{E}$ or $\vec{B}$ field. Horrible as it might seem, this might in principle *not* be true — we might have to "add up" such fields in some hopelessly more complicated way. But it didn't turn out that way in this universe. Lucky us!

17.2.4 The Lorentz Force

We can now put the second part of the procedure [calculating the *forces* on a test charge due to known FIELDS] into a very compact form combining both the electric and the magnetic forces into one equation. If a particle with charge q and mass m moves with velocity $\vec{v}$ in the combination of a uniform electric field $\vec{E}$ and a uniform magnetic field $\vec{B}$, the net force acting on the particle is the LORENTZ FORCE, which can be written (in one set of units)

$$\vec{F} \;=\; q\left(\vec{E} \,+\, \frac{\vec{v}}{c}\times\vec{B}\right), \tag{9}$$

where (for now) we can think of c as just some constant with units of velocity.

If $\vec{E}=0$ and $\vec{v}$ is *perpendicular* to $\vec{B}$, the Lorentz force is perpendicular to both $\vec{B}$ and the momentum $\vec{p}=m\vec{v}$. The force will deflect the momentum sideways, changing its direction but not its magnitude.[6] As $\vec{p}$ changes direction, $\vec{F}$ changes with it to remain ever perpendicular to the velocity — this is an automatic property of the cross product — and eventually the *orbit* of the particle closes back on itself to form a circle. In this way the magnetic field produces UNIFORM CIRCULAR MOTION with the plane of the circle perpendicular to both $\vec{v}$ and $\vec{B}$.

Using Newton's SECOND LAW and a general knowledge of circular motion, one can derive a formula for the *radius* of the circle (r) in terms of the *momentum* of the particle ($p=mv$), its *charge* (q) and the magnitude of the *magnetic field* (B). In "Gaussian units" (grams, centimeters, Gauss) the formula reads[7]

$$r = \frac{pc}{qB}. \tag{10}$$

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[6] A force perpendicular to the motion does no work on the particle and so does not change its kinetic energy or speed — recall the general qualitative features of CIRCULAR MOTION under the influence of a CENTRAL FORCE.

[7] In "practical" units the formula reads

$$r\,[\mathrm{cm}] \;=\; \frac{p\,[\mathrm{MeV/c}]}{0.3\;B\,[\mathrm{kG}]\;q\,[\text{electron charges}]}$$

where cm are (as usual) centimeters, MeV/c are millions of "electron volts" divided by the speed of light (believe it or not, a unit of momentum!) and kG ("kilogauss") are thousands of Gauss. I only mention this now because I will use it later on and because it illustrates the madness of electromagnetic units — see next Section!

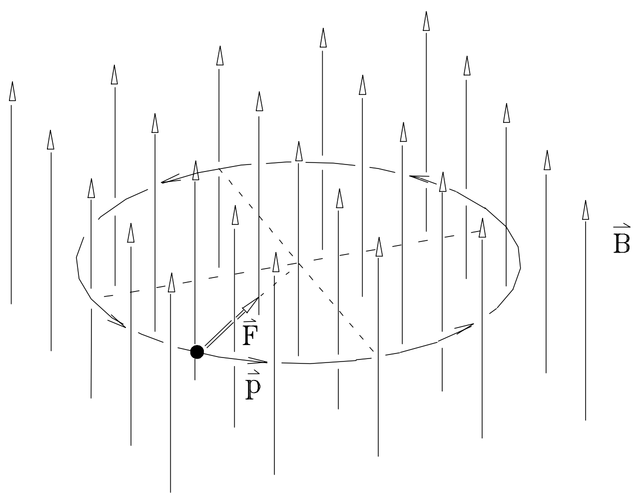

Figure 17.4 Path of a charged particle with momentum $\vec{p}$ in a uniform, static magnetic field $\vec{B}$ perpendicular to $\vec{p}$.

It is also interesting to picture qualitatively what will happen to the particle if an *electric* field $\vec{E}$ is then applied *parallel* to $\vec{B}$: since $\vec{E}$ accelerates the charge in the direction of $\vec{E}$, which is also the direction of $\vec{B}$, and since $\vec{B}$ only produces a force when the particle moves *perpendicular* to $\vec{B}$, in effect the "perpendicular part of the motion" is unchanged (circular motion) while the "parallel part" is unrestricted acceleration. The path in space followed by the particle will be a spiral with steadily increasing "pitch":

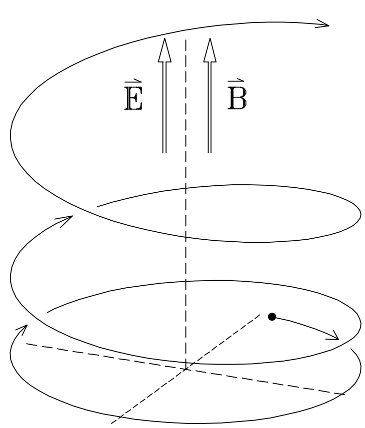

Figure 17.5 Path of a charged particle in *parallel* $\vec{E}$ and $\vec{B}$ fields.

17.2.5 "Field Lines" and Flux

In Fig. 17.4 the uniform magnetic field is pictured as a forest of little parallel arrows of equal length, equally spaced. Something like this is always necessary if we want to make a visual representation of $\vec{B}$, but it leaves a lot to be desired. For instance, a uniform magnetic field has the same magnitude and direction at *every* point in space, not just where the lines are drawn. Moreover, as we have seen, the magnetic *force*, if any, is *never* in the direction of the "lines of $\vec{B}$" but rather perpendicular to them, as shown in Fig. 17.4.

Nevertheless, the visual appeal of such a graphical representation in terms of "field lines" is so compelling that a whole description of $\mathcal{E}\&\mathcal{M}$ has been developed in terms of them. In that description one speaks of "lines per unit area" as a measure of the *strength* of an electric or magnetic field. The analogy is with *hydrodynamics*, the flow of incompressible fluids, in which we may actually see "lines" of fluid flow if we drop packets of dye in the water.

In fluid dynamics there is actually "stuff" flowing, a transfer of mass that has momentum and density. In that context one naturally thinks of the FLUX of material through imaginary surfaces perpendicular to the flow[8] and indeed $\vec{B}$ is sometimes referred to as the *magnetic flux per unit* (perpendicular) *area*.

By the same token, if "lines" of $\vec{B}$ pass through a surface of area A normal (perpendicular) to $\vec{B}$, then we can (and do) talk about the MAGNETIC FLUX Φ through the surface; Φ has units of magnetic field times area. If we want, we can turn this around and say that a magnetic field has units of *flux per unit area*.

Even though we rarely take this "lines of $\vec{B}$" business literally, it makes such a good image that we make constant use of it in handwaving arguments. Moreover, the concept of MAGNETIC FLUX is well ensconced in modern $\mathcal{E}\&\mathcal{M}$ terminology.

17.3 Potentials and Gradients

Recall from MECHANICS that if we move a particle a vector distance $d\vec{\ell}$ under the influence of a force $\vec{F}$, that force does $dW = \vec{F} \cdot d\vec{\ell}$ worth of *work* on the particle — which appears as *kinetic energy*. *Etc.* If the force is

[8]For instance, the flux of a river past a fixed point might be measured in gallons per minute per square meter of area perpendicular to the flow. A hydroelectric generator will intercept twice as many gallons per minute if it presents twice as large an area to the flow. And so on.

due to the action of an electric field $\vec{E}$ on a charge q, the work done is $dW = q\vec{E} \cdot d\vec{\ell}$. This work gets "stored up" as *potential energy* V as usual: $dV = -dW$. Just as we defined $\vec{E}$ as the *force per unit charge*, we now define the ELECTRIC POTENTIAL ϕ to be the *potential energy per unit charge*, *viz.*

$$dV = q\, d\phi \qquad \text{where} \qquad d\phi = -\vec{E} \cdot d\vec{\ell} \tag{11}$$

or, summing the contributions from all the infinitesimal elements $\vec{\ell}$ of a finite path through space in the presence of electric fields,[9]

$$\phi \equiv -\int \vec{E} \cdot d\vec{\ell} \tag{12}$$

When multiplied by q, ϕ gives the potential *energy* of the charge q in the electric field $\vec{E}$.

Just as we quickly adapted our formulation of MECHANICS to use *energy* (potential and kinetic) as a starting point instead of *force*, in $\mathcal{E}\&\mathcal{M}$ we usually find it easier to start from $\phi(\vec{r})$ as a function of position ($\vec{r}$) and derive $\vec{E}$ the same way we did in MECHANICS:

$$\vec{E} \equiv -\vec{\nabla}\phi \tag{13}$$

where, as before,[10]

$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{14}$$

The most important example is, of course, the electric potential due to a single "point charge" Q at the origin:

$$\phi(\vec{r}) = k_E \frac{Q}{r} \tag{15}$$

Note that $\phi(r) \to 0$ as $r \to 0$, as discussed in the previous footnote. This is a convenient convention. I will leave it as an exercise for the enthusiastic reader to show that

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}.$$

Electric potential is most commonly measured in *volts* (abbreviated V) after Count Volta, who made the first useful batteries. We often speak of the "voltage" of a battery or an appliance. [The latter does not ordinarily have any electric potential of its own, but it is designed to be *powered* by a certain "voltage." A light bulb would be a typical case in point.] The *volt* is actually such a familiar unit that eletric *field* is usually measured in the derivative unit, *volts per meter* (V/m). It really is time now to begin discussing *units* — what are those constants k_E and k_M, for instance? But first I have one last remark about *potentials*.

The electrostatic potential ϕ is often referred to as the SCALAR POTENTIAL, which immediately suggests that there must be such a thing as a VECTOR POTENTIAL too. Just so. The VECTOR POTENTIAL $\vec{A}$ is used to calculate the *magnetic* field $\vec{B}$ but not quite as simply as we get $\vec{E}$ from $\vec{\nabla}\phi$. In this case we have to take the "curl" of $\vec{A}$ to get $\vec{B}$:

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{16}$$

Never mind this now, but we will get back to it later.

17.4 Units

When Physicists are working out problems "formally" (that is, trying to understand "how things behave") they are usually only concerned with deriving a formula which describes the behaviour, not so much with getting "numbers" out of the formula. This is why we can tolerate so much confusion in the details of the alternate electromagnetic unit systems. We never actually calculate any "answers" that an engineer could use to build devices with; we simply derive a formula for such calculations, preferably in a form as free of specific units as possible, and leave the practical details up to the engineer (who may be us, later).

So I have left the unspecified "coupling constants" k_E and k_M undefined while we talked about the *qualitative* behaviour of electric and magnetic fields. Now we finally have to assign some *units* to all these weird quantities.

The history of *units* in $\mathcal{E}\&\mathcal{M}$ is a long horror story. It isn't even very entertaining, at least to my taste. Numerous textbooks provide excellent summaries of the different systems of units used in $\mathcal{E}\&\mathcal{M}$ [there are at least three!] but even when one understands perfectly there is not much satisfaction in it. Therefore I will provide only enough information on $\mathcal{E}\&\mathcal{M}$ units to define the unavoidable units one encounters in everyday modern life and to allow me to go on to the next subject.

As long as electric and magnetic fields are not both involved in the same problem, one can usually stick to familiar units expressed in a reasonably clear fashion. Let's discuss them one at a time.

[9] Note that, just as in the case of the mechanical potential energy V, the *zero* of ϕ is chosen arbitrarily at some point in space; we are really only sensitive to *differences* in potential. However, for a *point charge* it is conventional to choose an infinitely distant position as the zero of the electrostatic potential, so that $\phi(r)$ for a point charge Q is the work required to bring a unit test charge up to a distance r away from Q, starting at infinite distance.

[10] Remember the metaphor of $\vec{\nabla}\phi$ as the "slope" of a "hill" whose height is given by $\phi(\vec{r})$.

17.4.1 Electrical Units

I will give the old-fashioned version of this saga, in which one picks either VOLTS or COULOMBS as the "fundamental" unit and derives the rest from that. Today the AMPERE [A] is actually the most basic unit; it is *defined* to be the current required to flow in *both* of two "very long" parallel wires 1 *m* apart in order to give a *magnetic force per unit length* of exactly 2×10^{-7} *N/m* acting on each wire. No, I'm not kidding. Then the COULOMB [C] is defined as the electric charge that flows past any point in 1 *s* when a steady current of 1 A is maintained in a wire. *I.e.* we have 1 C = 1 A-*s*. Anyway, I will start with COULOMBS because it is more mnemonic.

Coulombs and Volts

As indicated in Eq. (2), electric *charge* is usually measured in COULOMBS (abbreviated C). If we take this as a fundamental unit, we can analyze the definition of the *volt* (V) by reference to Eq. (11): moving a charge of $q = 1$ C through an electric potential difference $\Delta\phi = 1$ V produces a potential energy difference of $\Delta V = 1$ J. Therefore

a VOLT is a *joule per coulomb*.

If we prefer to think of the *volt* as a more fundamental unit, we can turn this around and say that

a COULOMB is a *joule per volt*.

However, I think the former is a more comfortable definition.

Electron Volts

We can also take advantage of the fact that Nature supplies electric charges in integer multiples of a fixed quantity of charge[11] to define some more "natural" units. For instance, the electric charge of an electron is $-e$ [where e is the charge of a proton, defined in Eq. (2)]. An ELECTRON VOLT (eV) is the kinetic *energy* gained by an electron [or any other particle with the same size charge] when it is accelerated through a one volt (1 V) electric potential. Moving a charge of 1 C through a potential of 1 V takes 1 J of work (and will produce 1 J of kinetic energy), so we know immediately from Eq. (2) that

$$1 \text{ eV} \;=\; 1.60217733(49) \times 10^{-19} \text{ J} \qquad (17)$$

This is not much energy if you are a toaster, but for an electron (which is an *incredibly tiny* particle) it is enough to get it up to a velocity of 419.3828 km/s, which is 0.14% of the speed of light! Another way of looking at it is to recall that we can express *temperature* in energy units using Boltzmann's constant as a conversion factor. You can easily show for yourself that 1 eV is equivalent to a temperature of 11,604 degrees Kelvin or about 11,331°C. So in the microscopic world of electrons the eV is a pretty convenient (or "natural") unit. But not in the world of toasters and light bulbs. So let's get back to "conventional" units.

[11] This is what we mean when we say that charge is *quantized*.

Amperes

Electric currents (the rate at which charges pass a fixed point in a wire, for instance) have dimensions of *charge per unit time*. If the COULOMB is our chosen unit for electric charge and we retain our fondness for *seconds* as a time unit, then *current* must be measured in *coulombs per second*. We call these units AMPERES or Amps [abbreviated A] after a Frenchman named Ampère. Thus

$$1\,\text{A [AMPERE]} \;\equiv\; 1\,\text{C/s [COULOMB per second]} \qquad (18)$$

I have a problem with Amps. It makes about as much sense to give the coulomb per second its own name as it would to make up a name for meters per second. No one frets over the complexity of expressing speed in m/s or kph or whatever — in fact it serves as a good reminder that velocity is a rate of change of distance with time — but for some reason we feel obliged to give C/s their own name. Ah well, it is probably because all this electrical stuff is so weird.[12] Whatever the reason, we are stuck with them now!

The Coupling Constant

We are now ready to define our electrical "coupling constant" k_E. Referring to Eq.(15) we have

$$\phi\,[V] \;=\; k_E \, \frac{Q\,[C]}{r\,[m]}$$

which we can rearrange to read

$$k_E \;=\; \frac{\phi\,[V] \;\cdot\; r\,[m]}{Q\,[C]}$$

Thus k_E must have *dimensions* of {electric potential times distance per unit charge}; we can pick *units* of V-m/C to stick with convention. This still doesn't tell us the *value* of k_E. This must be *measured*. The result is

$$k_E \;=\; 8.98755\cdots \times 10^9 \text{ V-m/C} \qquad (19)$$

[12] And also, I suspect, because people were looking for a good way to honour the great Physicist Ampère and all the best units were already taken.

17.4.2 Magnetic Units

Gauss *vs.* **Tesla**

There are two "accepted" units for the magnetic field $\vec{B}$: GAUSS [abbreviated G] and TESLA [abbreviated T]. Needless to say, both are named after great $\mathcal{E}\&\mathcal{M}$ researchers. The former is handy when describing *weak* magnetic fields — for instance, the Earth's magnetic field is on the order of 1 G — but the unit that goes best with our selected electrical units (because it is defined in terms of meters and coulombs and seconds) is the TESLA. Fortunately the conversion factor is simple:

$$1 \text{ T} \equiv 10{,}000 \text{ G}.$$

The TESLA is also defined in terms of the WEBER [W] (named after guess whom), a conventional unit of magnetic *flux*. The definition is

$$1 \text{ T} \equiv 1 \text{ W/m}^2 \qquad \text{or} \qquad 1 \text{ W} = 1 \text{ T} \times 1 \text{ m}^2$$

if you're interested. So referring back to Eq. (8), we have

$$B\,[\text{TESLA}] \;=\; k_M \, \frac{I\,[\text{A}]\; d\ell[\text{m}]}{(r[\text{m}])^2}$$

which we can rearrange to read

$$k_M \;=\; \frac{B\,[\text{TESLA}]\;(r\,[\text{m}])^2}{I\,[\text{A}]\; d\ell\,[\text{m}]}$$

so that k_M must have *dimensions* of magnetic flux per unit current per unit length or *units* of W/A-m. Its *value* is again determined by experiment:

$$k_M \;=\; 10^7 \text{ W/A-m} \qquad (20)$$

I will leave it as an exercise for the student to plug these coupling constants back into the equations where they appear and show that everything is, though weird, dimensionally consistent.