Chapter 19

Ampère's law

With VECTOR CALCULUS firmly under our belts (?) we are now ready to tackle AMPÈRE'S LAW, right?

19.1 Integral Form

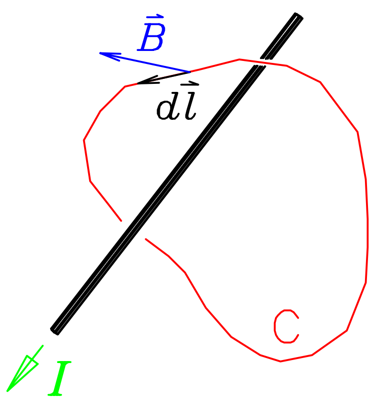

Figure 19.1 A wire carrying a current I passes through an arbitrary closed loop C, generating a magnetic field $\vec{\boldsymbol{B}}$ in the region around the wire. At every point on C there is a path element $d\vec{\boldsymbol{\ell}}$ in the direction around the loop corresponding to the direction the fingers of your right hand would point if you grabbed the wire with your thumb pointing along the current, and a magnetic field $\vec{\boldsymbol{B}}$ in some direction (not necessarily the same direction as $d\vec{\boldsymbol{\ell}}$).

If at each step $d\vec{\boldsymbol{\ell}}$ around the path C in Fig. 19.1 we find the component of the magnetic field $\vec{\boldsymbol{B}}$ in the direction of $d\vec{\boldsymbol{\ell}}$, multiply the two, and add up all the results for the whole loop, we get the *integral form* of AMPÈRE'S LAW:

$$\boxed{\oint_C \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 I} \tag{1}$$

where $\mu_0 = 4\pi \times 10^{-7}$ Webers/(Amp·m) [or Newtons/Amp^2, or Henries/m, or Tesla·m/Amp, or Volt·s/(Amp·m)] is the PERMEABILITY OF FREE SPACE.[1]

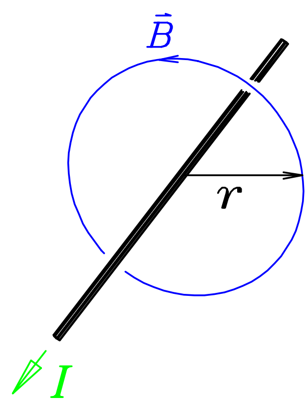

Figure 19.2 By symmetry, a wire carrying a current I generates a magnetic field $\vec{\boldsymbol{B}}$ that forms circular loops centered on the wire at every radius r.

In cases where the direction of $\vec{\boldsymbol{B}}$ at every point along the path C is not known, this form is pretty useless for practical calculations. But the LAW OF BIOT & SAVART tells us that the contribution to $\vec{\boldsymbol{B}}$ from each element of current is always perpendicular to the current and proportional to the inverse square of the distance

[1] What can I say? Electromagnetic units are *weird*!

from that current element; so SYMMETRY demands that a "line of $\vec{\boldsymbol{B}}$" forms a *circular* loop centered on the wire, as shown in Fig. 19.2, and that its magnitude is the same everywhere around that loop. So we simply pick such a loop of radius r as our path C, and the path integral on the left side of Eq. (1) becomes just

$$\oint_C \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = 2\pi r B$$

giving

$$B(r) = \frac{\mu_0 I}{2\pi r} \tag{2}$$

as we found in the earlier Exercise.

19.2 Differential Form

We can apply STOKES' THEOREM to the integral in Eq. (1) to get

$$\iint_A (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}) \cdot d\vec{\boldsymbol{S}} = \mu_0 I$$

and note that

$$I = \iint_A \vec{\boldsymbol{J}} \cdot d\vec{\boldsymbol{S}}$$

on the same surface A bounded by the path C in Fig. 19.1. Therefore the integrands of the two surface integrals must be equal:

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}}$$

or

$$\boxed{\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}}} \tag{3}$$

where $\vec{\boldsymbol{J}}$ is the CURRENT DENSITY and we have defined

$$\vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{H}} \tag{4}$$

in free space. (In magnetic materials $\vec{\boldsymbol{B}} = \mu \vec{\boldsymbol{H}}$ where μ is the magnetic permeability of the material.) Equation (3) expresses the relationship between the CURRENT DENSITY and the curl of the magnetic field *at any point in space*. This is pretty cool too!

But we have left something out....

19.3 Displacement Current

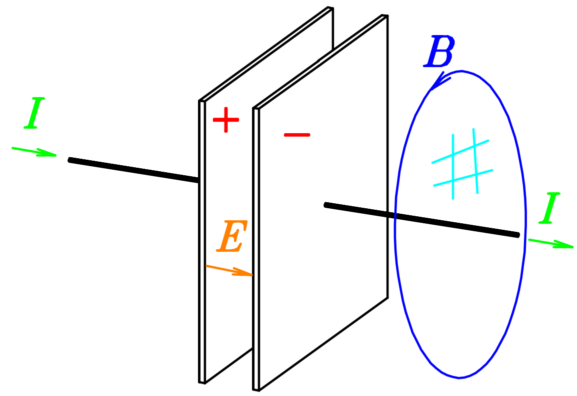

Figure 19.3 A capacitor consists of two adjacent plates of conductor separated by an insulator (*e.g.* air). The plates are initially uncharged. If a current begins flowing onto the left plate, it starts to accumulate a positive charge; this attracts negative charges on the light plate, which must come down the wire on the right (from "elsewhere"). negative charges flowing to the left constitutes a positive current to the right, so the current appears (at least initially) to *pass through* the capacitor, even though one plate is isolated from the other. The surface charges produce an electric field $\vec{\boldsymbol{E}}$ between the plates (and a voltage $V = Ed$ where d is the distance between the plates). Since E is proportional to the accumulated charge on the plate, $\partial E/\partial t \propto I$.

James Maxwell reasoned that an application of the integral form of AMPÈRE'S LAW to find the magnetic field encircling the wire far from the capacitor was supposed to work for **any** surface bounded by the path over which the line integral of $\vec{\boldsymbol{B}}$ is evaluated, it should give the same answer whether that surface is "punctured" by the current or not.

Visualize, if you will, a soap bubble across the blue loop shown in Fig. 19.3. The current I clearly "punctures" that surface. Now blow to the left through the blue loop and imagine that the right plate of the capacitor somehow fails to pop the resultant bubble, so that the surface bounded by the blue loop now passes *between* the capacitor plates, where there are no moving charges. What gives?

Let's review the electric field between two capacitor plates: By GAUSS' LAW it's constant far from the edges, points from the + plate to the − plate, and has a magnitude $E = \sigma/\epsilon_0$, where $\sigma = Q/A$ (the charge on one plate divided by the area of the plate). Thus $\epsilon_0 E = D = Q/A$ and taking the time derivative gives

$$A \cdot \frac{\partial D}{\partial t} = \frac{\partial Q}{\partial t} \equiv I.$$

But since $\vec{\boldsymbol{D}}$ is constant over the area A and zero outside the capacitor, we can write this as

$$\iint_A \frac{\partial \vec{\boldsymbol{D}}}{\partial t} \cdot d\vec{\boldsymbol{S}} = I.$$

That is, a changing electric field is equivalent to an actual current.

Maxwell called this surface integral of the changing electric field a DISPLACEMENT CURRENT after the name of $\vec{\boldsymbol{D}}$ (the "electric displacement"). It turns out (with a little more rigorous derivation) to hold equally well for less simple geometries, giving us MAXWELL'S EXTENSION OF AMPÈRE'S LAW,

$$\boxed{\oint_C \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{\ell}} = \iint_A \left(\vec{\boldsymbol{J}} + \frac{\partial \vec{\boldsymbol{D}}}{\partial t} \right) \cdot d\vec{\boldsymbol{S}}.} \qquad (5)$$

which is equivalent to the differential version,

$$\boxed{\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} + \frac{\partial \vec{\boldsymbol{D}}}{\partial t}} \qquad (6)$$