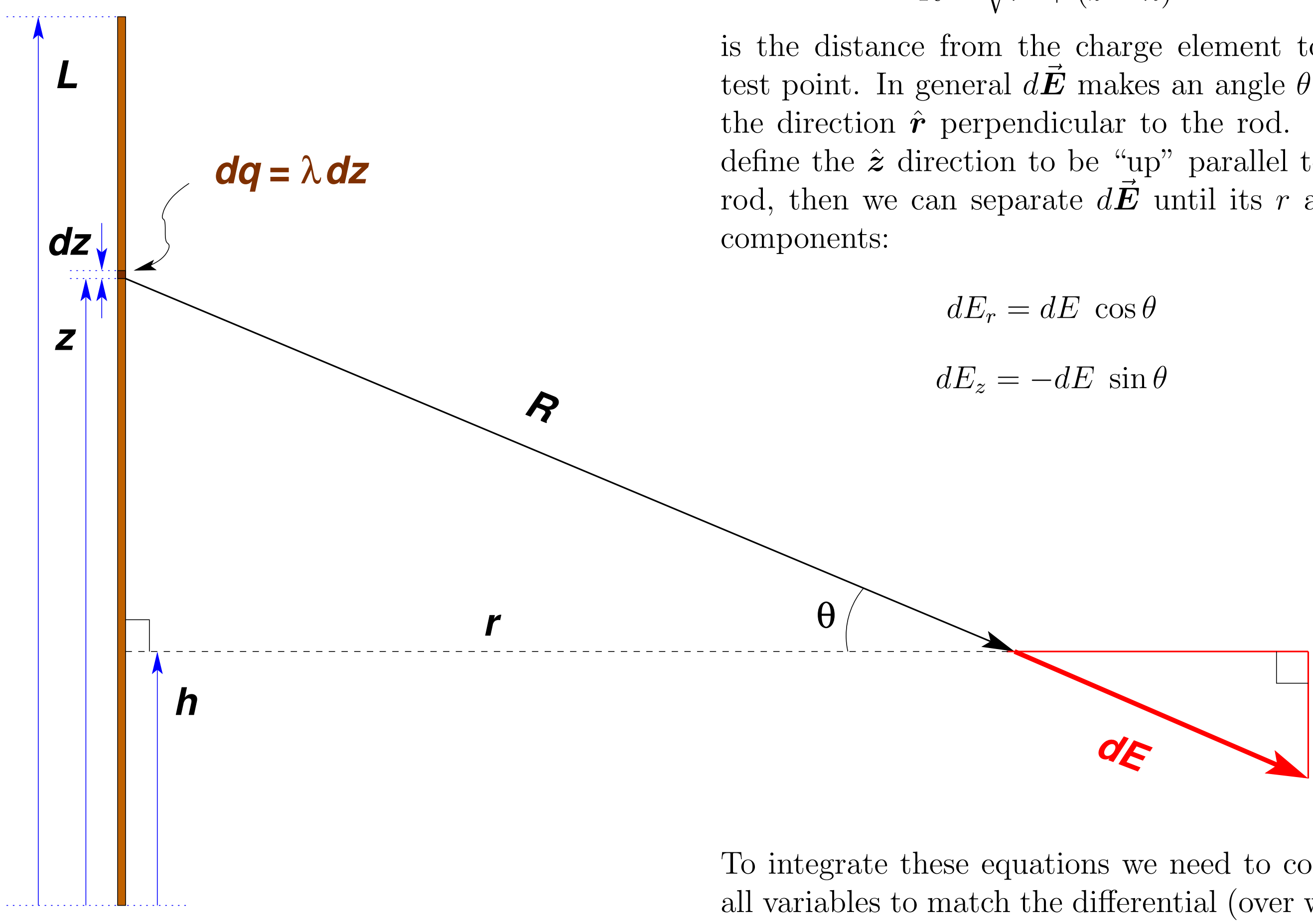
# Rod of Charge

As an exercise in the "brute force" integration of COULOMB'S LAW (unavoidable in most cases), here is one way to find the electric field due to a uniformly charged, skinny rod of finite length $L$:

If the total charge $Q$ is uniformly distributed along the rod, then the charge per unit length is

$$\lambda = \frac{Q}{L} \,. \tag{1}$$




We want to evaluate the electric field $\vec{\boldsymbol{E}}$ at an arbitrary "test point" in space. Such a point can be characterized completely by its perpendicular distance $r$ from the rod and its distance $h$ up from the bottom end of the rod, measured parallel to the rod, as shown. We choose to look at the rod and the point in their common plane.

Then we pick an arbitrarily position a distance $z$ up from the bottom end of the rod, as shown. A small slice of the rod (width $dz$) at that position contains a "charge element" $dq = \lambda\, dz$ which contributes $d\vec{\boldsymbol{E}}$ to the electric field vector $\vec{\boldsymbol{E}}$ at the test point. Coulomb's Law says that $d\vec{\boldsymbol{E}}$ points away from the charge element (assuming positive charge) and has a magnitude

$$dE = \frac{k_E \lambda\, dz}{R^2} \tag{2}$$

where

$$R = \sqrt{r^2 + (z-h)^2} \tag{3}$$

is the distance from the charge element to the test point. In general $d\vec{\boldsymbol{E}}$ makes an angle $\theta$ with the direction $\hat{\boldsymbol{r}}$ perpendicular to the rod. If we define the $\hat{\boldsymbol{z}}$ direction to be "up" parallel to the rod, then we can separate $d\vec{\boldsymbol{E}}$ until its $r$ and $z$ components:

$$dE_r = dE\ \cos\theta \tag{4}$$

$$dE_z = -dE\ \sin\theta \tag{5}$$

To integrate these equations we need to convert all variables to match the differential (over which we integrate). We could use Eq. (3) to express $R$ in terms of $z$ (where $r$ and $h$ are constants) and use

$$\cos\theta = \frac{r}{R} \tag{6}$$

$$\sin\theta = \frac{(z-h)}{R} \tag{7}$$

but this would leave us with integrals that cannot be solved by inspection.

If we want to solve this problem without reference to external aids (like tables of integrals), it is better to convert into angles and trigonometric functions as follows:

Equation (6) can be rewritten

$$\frac{1}{R^2} = \frac{\cos^2\theta}{r^2} \tag{8}$$

and since

$$z - h = r\tan\theta \ , \tag{9}$$

giving

$$dz = r\sec^2\theta \ d\theta = \frac{r\,d\theta}{\cos^2\theta} \ , \tag{10}$$

we can write Eq. (2) as

$$dE = k_E\lambda\left(\frac{\cos^2\theta}{r^2}\right)\left(\frac{r\,d\theta}{\cos^2\theta}\right) = \frac{k_E\lambda}{r}\,d\theta \tag{11}$$

and from that, Eqs. (4) and (5), respectively, as

$$dE_r = \frac{k_E\lambda}{r}\cos\theta \ d\theta = \frac{k_E\lambda}{r}\,du \tag{12}$$

where $u \equiv \sin\theta$, and

$$dE_z = -\frac{k_E\lambda}{r}\sin\theta \ d\theta = \frac{k_E\lambda}{r}\,dv \tag{13}$$

where $v \equiv \cos\theta$.

Integrating these differentials is trivial; we are left with just the differences between $u$ (or $v$) at the limits of integration (the top and bottom of the rod):

$$E_r = \frac{k_E\lambda}{r}\left[\frac{(L-h)}{\sqrt{r^2+(L-h)^2}} + \frac{h}{\sqrt{r^2+h^2}}\right] \tag{14}$$

(note that $u$ is negative at the bottom) and

$$E_z = k_E\lambda\left[\frac{1}{\sqrt{r^2+(L-h)^2}} - \frac{1}{\sqrt{r^2+h^2}}\right] \tag{15}$$

These equations express a completely general solution to this problem.

Let's check to see what these give for the field directly out from the *midpoint* of the rod — *i.e.* for $h = L/2$:

$$\begin{aligned} E_r &= \frac{k_E\lambda}{r}\left[\frac{L/2}{\sqrt{r^2+L^2/4}} + \frac{L/2}{\sqrt{r^2+L^2/4}}\right] \\ &= \frac{k_E\lambda}{r}\,\frac{L}{\sqrt{r^2+L^2/4}} \end{aligned} \tag{16}$$

and

$$\begin{aligned} E_z &= k_E\lambda\left[\frac{1}{\sqrt{r^2+L^2/4}} - \frac{1}{\sqrt{r^2+L^2/4}}\right] \\ &= 0 \ . \end{aligned} \tag{17}$$

Let's also check to see what we get for $E_r$ (at the midpoint) very far from the rod ($r \gg L$):

$$E_r \underset{r\to\infty}{\longrightarrow} \frac{k_E\lambda L}{r^2} = \frac{k_E Q}{r^2} \tag{18}$$

(*i.e.* Coulomb's Law) √

and very close to the rod ($r \ll L$):

$$E_r \underset{r\to 0}{\longrightarrow} \frac{2k_E\lambda}{r} \ . \tag{19}$$

The last result can be used as the field due to an *infinitely long* uniform line of charge. But there is a much easier way to obtain it. . . .