The University of British Columbia

# Physics 401 Assignment # 1: REVIEW

Wed. 04 Jan. 2006 — finish by Wed. 11 Jan.

This first assignment is just review, to make sure you haven't forgotten (or can quickly recall) what you learned in PHYS 301/354 (or earlier) about the E&M covered in the first 7 chapters of our textbook: David Griffiths, *"Introduction to Electrodynamics"*.

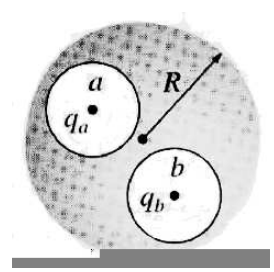
**1. MAXWELL'S EQUATIONS:**

($a$) Starting with Maxwell's equations in differential form, derive Maxwell's equations in integral form.

($b$) Starting with Maxwell's generalization of Ampère's Law, $\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$, derive the continuity equation, $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$, which is the mathematical expression of charge conservation.

($c$) Starting with Maxwell's equations in free space ($\vec{J} = 0,\ \rho = 0$), show that $\vec{E}$ and $\vec{B}$ each satisfy a wave equation. What is the speed of propagation of the resulting wave in each case?

**2. CHARGED CONDUCTORS:** Two spherical cavities, of radii $a$ and $b$, are hollowed out from the interior of a solid neutral conducting sphere of radius $R$, as shown in the figure. There are charges $q_a$ and $q_b$ at the centres of the respective cavities.




($a$) What is the electric field in the solid (shaded) conducting material?

($b$) Find the surface charges $\sigma_a$, $\sigma_b$ and $\sigma_R$ at the respective surfaces.

($c$) What is the electric field outside the conductor at a distance $r > R$ from the centre of the large sphere?

($d$) What are the electric fields inside cavities $a$ and $b$?

($e$) What are the forces on $q_a$ and $q_b$?

($f$) If a third charge $q_c$ were brought near the conductor, which (if any) would change:

($i$) $\sigma_a$?

($ii$) $\sigma_b$?

($ii$) $\sigma_R$?

($iv$) The electric fields inside cavities $a$ and $b$?

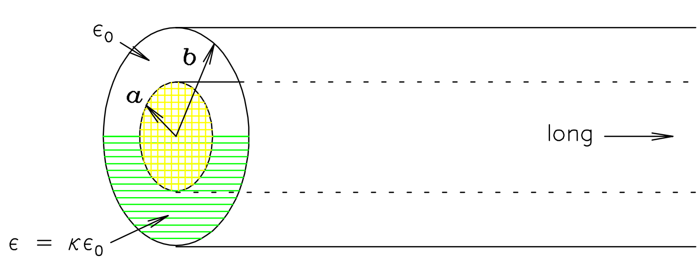
($v$) The electric field outside the conductor?

**3. COAXIAL CAPACITOR:** A capacitor is constructed of two very long concentric cylindrical conductors with their common axis horizontal, as shown in the diagram. The space between them is exactly half filled with a linear dielectric liquid with dielectric constant $\kappa$.

($a$) Show that the electric field is radial and is the same in the dielectric half as in the vacuum half of the capacitor.

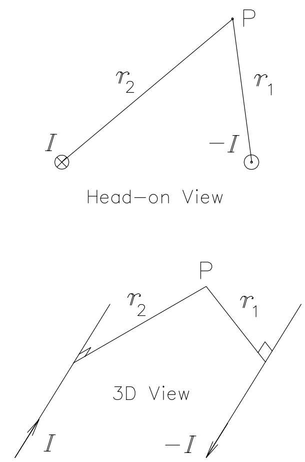
($b$) Deduce the capacitance per unit length of this coaxial capacitor.

($c$) If the conductors carry free charges per unit length $\pm\lambda$, find the polarization $\vec{P}$ in the dielectric at any point a distance $r$ from the central axis, in terms of $\epsilon_0, \kappa, \lambda$ and $r$.



**4. LINEAR CURRENTS:**

Two very long parallel wires carry equal currents $\pm I$ in opposite directions, as illustrated in the figure. Take the $\hat{\boldsymbol{z}}$ direction to be out of the page, in the direction of the current in wire 1. The field point P is located a distance $r_1$ from wire 1 and a distance $r_2$ from wire 2, as shown.

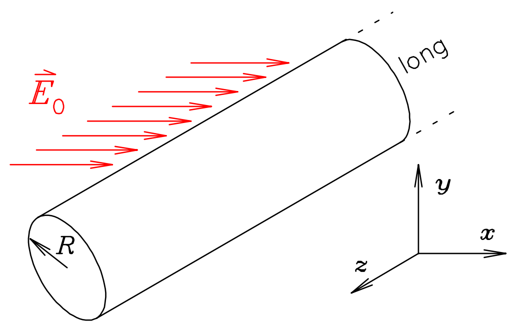



($a$) Consider each wire separately and indicate the direction of the vector potential $\vec{\boldsymbol{A}}$ in each case.

($b$) Show that the vector potential $\vec{\boldsymbol{A}}$ at the point P is given by: $\vec{\boldsymbol{A}} = \frac{\mu_0 I}{2\pi} \ln\left(\frac{r_2}{r_1}\right)\hat{\boldsymbol{z}}$

($c$) Show that the result in part ($b$) is consistent with that obtained using Ampère's Law.

**5. LAPLACE'S EQUATION:** Consider an infinitely long metal pipe, of radius $R$, which is placed at right angles to an otherwise uniform electric field $\vec{\boldsymbol{E}}_0 = E_0\hat{\boldsymbol{x}}$.




($a$) What is the "uniqueness theorem" and why would you want to use it to solve for the electric potential $V$?

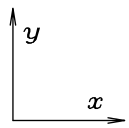
($b$) What are the boundary conditions on the electric potential $V$?

($c$) Solve Laplace's equation for the potential $V$ outside the long metal pipe. You should obtain: $V(r,\theta) = E_0 r\left(\frac{R^2}{r^2} - 1\right)\cos\theta$ .

Hint: Note that this situation has cylindrical symmetry (not spherical!), with no $z$ dependence, and hence simplifies to a 2-D plane polar problem.

# Solutions to Laplace's Equation: $\nabla^2 V = 0$

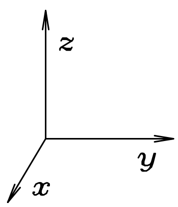
## 2D Cartesian:




$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V(x,y) = \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} y \\ 1 \end{matrix}\right\} + \left.\begin{matrix} e^{kx} \\ e^{-kx} \end{matrix}\right\} \left.\begin{matrix} \cos ky \\ \sin ky \end{matrix}\right\} \quad + \text{ permutations } (x \leftrightarrow y).$$

## 3D Cartesian:

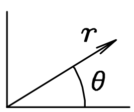



$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(x,y,z) = \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} y \\ 1 \end{matrix}\right\} \left.\begin{matrix} z \\ 1 \end{matrix}\right\} + \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} \cos py \\ \sin py \end{matrix}\right\} \left.\begin{matrix} e^{qz} \\ e^{-qz} \end{matrix}\right\} + \left.\begin{matrix} e^{px} \\ e^{-px} \end{matrix}\right\} \left.\begin{matrix} \cos qy \\ \sin qy \end{matrix}\right\} \left.\begin{matrix} \cos \sqrt{p^2 - q^2}\, z \\ \sin \sqrt{p^2 - q^2}\, z \end{matrix}\right\}$$

$+$ all permutations $\{x, y, z\}$.

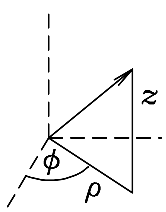
## 2D Plane Polar:




$$\nabla^2 V \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

$$V(r,\theta) = \left.\begin{matrix} \ln r \\ 1 \end{matrix}\right\} + \left.\begin{matrix} r^n \\ r^{-}n \end{matrix}\right\} \left.\begin{matrix} \cos n\theta \\ \sin n\theta \end{matrix}\right\}$$

## 3D Cylindrical:

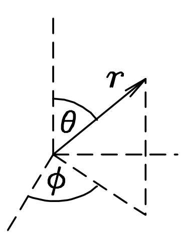



$$\nabla^2 V \equiv \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(\rho,\phi,z) = \left.\begin{matrix} J_n(k\rho) \\ N_n(k\rho) \end{matrix}\right\} \left.\begin{matrix} \cos n\phi \\ \sin n\phi \end{matrix}\right\} \left.\begin{matrix} e^{kz} \\ e^{-kz} \end{matrix}\right\}$$

where $J_n(k\rho) \rightarrow$ Bessel functions and $N_n(k\rho) \rightarrow$ Neumann functions.

## 3D Spherical:




$$\nabla^2 V \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$V(r,\theta,\phi) = \left.\begin{matrix} r^\ell \\ r^{-(\ell+1)} \end{matrix}\right\} \left.\begin{matrix} P_\ell^m(\cos\theta) \\ Q_\ell^m(\cos\theta) \end{matrix}\right\} \left.\begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix}\right\}$$

where $P_\ell^m(\cos\theta)$ are associated Legendre polynomials

and $Q_\ell^m(\cos\theta)$ are associated Legendre polynomials of the second kind.

*If axial symmetry* then $V(r,\theta,\phi) = \left.\begin{matrix} r^\ell \\ r^{-(\ell+1)} \end{matrix}\right\} \left.\begin{matrix} P_\ell(\cos\theta) \\ Q_\ell(\cos\theta) \end{matrix}\right\}$

where $P_\ell(\cos\theta)$ are Legendre polynomials and $Q_\ell(\cos\theta)$ are Legendre polynomials of the second kind.

Match **linear combinations** of the forms above to the appropriate **boundary conditions** imposed by (*e.g.*) conducting surfaces (equipotentials) and any requirements that $V \underset{r\to\infty}{\longrightarrow} 0$ *etc.*